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Theoretical studies of smectic C liquid crystals confined in a wedge

Stability considerations and Frederiks transitions

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A general deformation of a smectic C liquid crystal is composed of five different distortions, each of which can be made independently. Here we show that to each of these distortions we can assign a simple vector operator. Use of these five basis operators enables us to write down the elastic free energy density as a quadratic form consisting of nine terms. We also discuss how the nine elastic constants defined by the elastic energy expansion must fulfil certain restrictions in terms of inequalities and a specific tilt angle dependence. Assuming the smectic layers to be incompressible, we examine how certain arrangements of the smectic layers can be stable due to an interplay between the incompressibility condition and the boundary conditions which we impose on the director. One such stable configuration is the wedge, where the smectic layers form parts of concentric cylinders with the common axis coinciding with the centre of the wedge. For such a system we discuss the different director configurations which can be achieved and their stability. We also discuss the possibility of inducing Frederiks transitions for some of these configurations and calculate the corresponding thresholds, thereby demonstrating the design of an experiment which would make it possible to measure those elastic constants which are related to the deformations of the smectic layers, constants which are normally difficult to determine experimentally.

1. Introduction

Liquid-crystalline systems consist of elongated molecules for which the long molecular axes locally adopt one common direction in space. This direction is generally described by a unit vector **n**, commonly called the director. Furthermore, smectic C liquid crystals are layered structures for which the director makes an angle θ with respect to the layer normal. In order to describe an S_C liquid crystal a unit vector **a** defining the layer normal and a unit vector **c** are normally introduced, the c-director defining the projection of the director onto the smectic planes. We will also find it convenient to define an angle ϕ which describes the orientation of the c-director with

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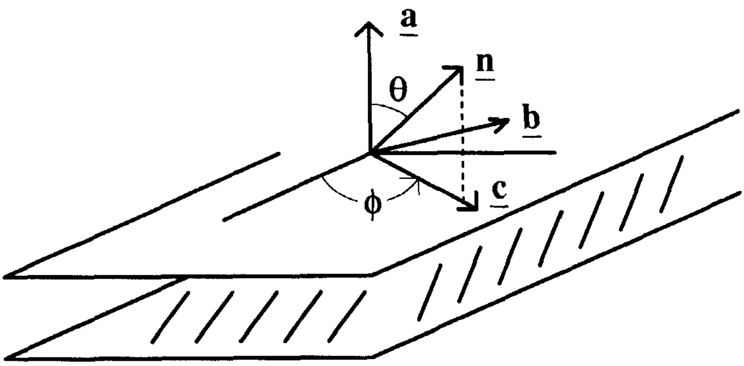

Figure 1. Notation used in the present work. The average molecular direction, i.e. the director, is given by a unit vector **n** making an angle θ with the layer normal **a**. The c-director, being a unit vector parallel to the projection of the director into the smectic planes, is denoted by **c** and can be described by the angle ϕ. The unit vector **b**, which is also confined to lie within the smectic planes, is defined by the relation $\mathbf{b}=\mathbf{a}\times\mathbf{c}$.

respect to some reference direction within the smectic planes. For mathematical convenience we also introduce a unit vector **b** (the C_2 axis of symmetry) which lies within the smectic planes and is perpendicular to **c**. In order to obtain an unambiguous definition of the sign of **b** we introduce **b** as

$$\mathbf{b}=\mathbf{a}\times\mathbf{c}. \tag{1}$$

The notations defined here are pictured in figure 1.

An indisturbed, relaxed S_C liquid crystal will form flat layers and have a c-director which is uniform in space, that is, it is described by the two vectors **a** and **c** being constant. If we apply some conflicting boundary conditions or an external field to the system, this state can be perturbed thus causing an elastic deformation. There are two principally different ways of imposing elastic deformations on a S_C liquid crystal (we only consider the case of constant layer thickness). First, we can deform the smectic layers keeping the c-director constant with respect to the layers. Secondly, we can rotate the c-director around the layer normal keeping the layers unchanged. To each of these two types of deformations a set of elastic constants is associated. In the notation we will use, a set of elastic constants A_i is assigned to describe the layer deformations while another set B_i is assigned to describe c-director rotations. The elastic free energy density also contains coupling terms between these two types of deformations and these will be associated with the elastic constants C_i.

While, for a given layering of the smectic planes the only requirement for the c-director is that it is a continuous function in space (disregarding the possibility of formation of disclinations), the layer normal **a** has to fulfil a more severe constraint. If we are dealing with a system of constant layer thickness which is also assumed free from dislocations the relation

$$\nabla\times\mathbf{a}=0 \tag{2}$$

must hold [1]. This drastically restricts the possible ways to arrange the smectic layers in space. The simplest arrangements of the smectic planes which fulfil equation (2) other than undistorted planes are layers which form concentric cylinders or spheres or parts thereof. More complex configurations satisfying equation (2) consist of the focal conics and the Dupin cyclides [2, 3]. Once the layers have formed one of the allowed configurations, we may find them to be stable in the sense that all possible perturbations of **a** compatible with the given boundary conditions will violate the constraint $\nabla \times \mathbf{a} = 0$. We use the term geometrically stable to describe such configurations. The only way to perturb the layers of such a configuration under the constraint (2) would be to compress or dilate the layers, that is, to introduce a deformation which is associated with a very large elastic energy [2]. For this reason such compressions or dilations will be suppressed.

When measuring the elastic constants one of two different methods is normally used. One way is to prepare the system in an equilibrium configuration and then change some of the external conditions (often by means of an electric or magnetic field) in order to find some threshold for which the configuration of the system transforms into a new state of lower energy. We can also perform dynamic experiments where frequently the response time of the system is measured by applying an alternating electric field. However, due to the facts just discussed, all reported measurements of the elastic constants of S_C liquid crystals only consider c-director distortions and no values of the constants A_i (or C_i) have been reported [4–7]. This results from the fact that for the configurations used in previous measurements the layers are geometrically stable in the sense defined and layer deformations are suppressed if we do not allow for compression or dilation of the layers.

In this paper we will demonstrate how to find a geometrical arrangement which makes it possible to perform a Frederiks transition type of experiment for which the threshold will depend partly on the A_i constants, thus allowing us to design an experiment for measuring them. The outline of the paper is as follows. In § 2 we review the elastic free energy density of S_C liquid crystals and define the set of elastic constants we use for solving the proposed problem. We also present a suitable set of vector operators through which the expression of the elastic free energy density becomes fairly simple. The temperature dependence of the elastic constants enters primarily through the tilt angle dependence and we discuss how this can be deduced from symmetry arguments. In § 3 we discuss some possible equilibrium configurations for a S_C liquid crystal contained in a wedge. We also discuss the stability of these configurations. By assuming an electric field to be applied over the wedge in an appropriate way, we then derive, in § 4, the threshold of a Frederiks transition which the system can exhibit. From this expression we will see that even if the configurations of the system which we study are geometrically stable we can deduce some information about the elastic constants connected with the layer deformations (the A_i constants) by performing the proposed experiment.

2. The elastic free energy density of the S_C phase

2.1. *An expression for the elastic energy in terms of vector operators*

The elastic free energy density of S_C liquid crystals has been discussed by a number of authors [1, 8, 9]. A common feature of the different approaches is that nine elastic constants are required to write down the complete bulk elastic energy of the system. In this section we review these approaches and show how we can write down a set of vector operators suitable to describe the elastic deformations which can be exhibited by the system. To describe a general deformed state of the system we have to specify the spatial variations of the two unit vectors **a** and **c** subject to the constraints $\mathbf{a} \cdot \mathbf{c} = 0$ and $\nabla \times \mathbf{a} = 0$. However, by introducing the vector **b** as in equation (1) it is clear that if we specify the spatial dependence of any two of the three vectors **a**, **b** and **c** we have defined the configuration of the system unambiguously. At first sight it seems to be more natural to formulate an elastic theory in terms of the two vectors **a** and **c**, which correspond to the two observable physical quantities used to define the system. We will, however, show that if we want to achieve a simple mathematical formulation of the problem which we consider it is preferable to write down the elastic energy in terms of **b** and **c**.

The first formulation of the S_C elastic energy was given by the Orsay Group [1]. In their formulation they introduce a local coordinate frame oriented in such a way that the *z* axis coincides with the layer normal while the c-director is always parallel to the *x* axis (cf. figure 2 (*a*)). The elastic deformations are then described by a rotation vector $\mathbf{\Omega}$. Rotations corresponding to the components Ω_x and Ω_y will always affect the direction of the layer normal **a** and thus correspond to layer deformations. Rotations corresponding to Ω_z will on the other hand leave **a** unaffected thus corresponding to rotations of the c-director. Although the $\mathbf{\Omega}$ notation of the Orsay Group permits a very clear physical interpretation it is, unfortunately, not well suited as a starting point for doing calculations. We prefer therefore to express the elastic deformations of the system in terms of a set of vector operators which are allowed to act on the two vectors **b** and **c**. In figures 2(*b*)–(*f*) we show the five different eigen deformations the system can undergo. We also give corresponding operators in terms of our **b** – **c** formulation and, as a comparison, in terms of that of the Orsay Group. For simplicity, in the figure we only show one smectic plane as seen from above and the corresponding c-director as it varies within this plane. First, we can imagine two different ways of bending the planes, still keeping the c-director unchanged with respect to the plane. In figure 2(*b*) the smectic plane bends in such a way that the layer normal **a** changes as we go along the direction of the c-director. This corresponds to $\mathbf{b}\cdot\nabla\times\mathbf{c}\neq0$ or $\partial\Omega_y/\partial x\neq0$ in the notation of the Orsay Group. If, on the other hand, we bend the planes in such a way that **a** changes as we go along the direction perpendicular to the c-director, we have the situation pictured in figure 2 (*c*). This corresponds to $\mathbf{c}\cdot\nabla\times\mathbf{b}\neq0$ $(\partial\Omega_x/\partial y\neq0)$. Keeping the layers planar we can rotate the c-director either as we move parallel (see figure 2 (*d*)) or perpendicular (see figure 2(*e*)) to it. This corresponds to $\nabla\cdot\mathbf{b}\neq0$ $(\partial\Omega_z/\partial x\neq0)$ and $\nabla\cdot\mathbf{c}\neq0$ $(\partial\Omega_z/\partial y\neq0)$, respectively. Finally, if we keep the c-director constant in each plane, rotating it as we go along the layer normal we have the situation pictured in figure 2 (*f*) where $\frac{1}{2}(\mathbf{b}\cdot\nabla\times\mathbf{b}+\mathbf{c}\cdot\nabla\times\mathbf{c})\neq0$ $(\partial\Omega_z/\partial z\neq0)$. The elastic free energy density can now be expressed in terms of these five deformations and of some cross products between them. The symmetry of the system will limit the number of corresponding terms since under the symmetry operations of the S_C phase the three terms $\nabla\cdot\mathbf{c}$, $\mathbf{b}\cdot\nabla\times\mathbf{c}$ and $\mathbf{c}\cdot\nabla\times\mathbf{b}$ transform in one way while $\nabla\cdot\mathbf{b}$ and $\frac{1}{2}(\mathbf{b}\cdot\nabla\times\mathbf{b}+\mathbf{c}\cdot\nabla\times\mathbf{c})$ transform in another. As the elastic energy of the system must be unaltered under a symmetry operation we can only construct cross coupling terms by combining operators within each of these two groups. The elastic energy of the system can thus be written as

$$\begin{aligned} w={}&\tfrac{1}{2}A_{12}(\mathbf{b}\cdot\nabla\times\mathbf{c})^2+\tfrac{1}{2}A_{21}(\mathbf{c}\cdot\nabla\times\mathbf{b})^2+A_{11}(\mathbf{b}\cdot\nabla\times\mathbf{c})(\mathbf{c}\cdot\nabla\times\mathbf{b})\\ &+\tfrac{1}{2}B_1(\nabla\cdot\mathbf{b})^2+\tfrac{1}{2}B_2(\nabla\cdot\mathbf{c})^2+\tfrac{1}{2}B_3[\tfrac{1}{2}(\mathbf{b}\cdot\nabla\times\mathbf{b}+\mathbf{c}\cdot\nabla\times\mathbf{c})]^2\\ &+B_{13}(\nabla\cdot\mathbf{b})[\tfrac{1}{2}(\mathbf{b}\cdot\nabla\times\mathbf{b}+\mathbf{c}\cdot\nabla\times\mathbf{c})]\\ &+C_1(\nabla\cdot\mathbf{c})(\mathbf{b}\cdot\nabla\times\mathbf{c})+C_2(\nabla\cdot\mathbf{c})(\mathbf{c}\cdot\nabla\times\mathbf{b}). \end{aligned} \qquad (3a)$$

By this equation we have defined the nine elastic constants of the S_C phase. An expression in the same spirit as equation (3*a*) has been derived by Rapini [10]. We

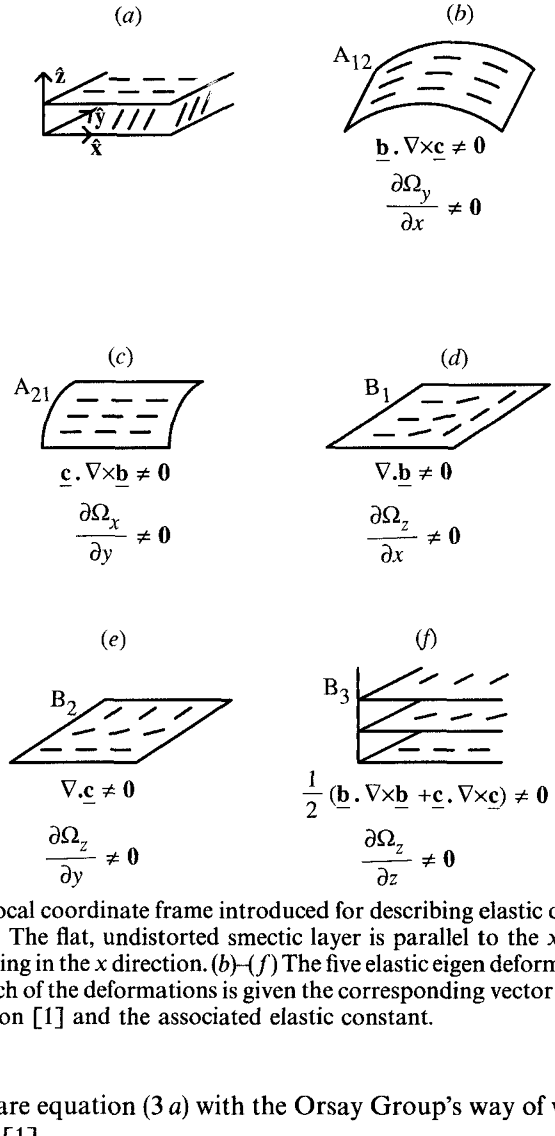

Figure 2. (*a*) The local coordinate frame introduced for describing elastic deformations of a S_C liquid crystal. The flat, undistorted smectic layer is parallel to the xy plane with the c-director pointing in the x direction. (*b*)–(*f*) The five elastic eigen deformations of a S_C liquid crystal. To each of the deformations is given the corresponding vector operator, the Orsay Group notation [1] and the associated elastic constant.

should also compare equation (3 *a*) with the Orsay Group's way of writing the S_C free energy density as [1]

$$\begin{aligned} w^{\text{Orsay}} &= \tfrac{1}{2}A_{12}^{0}\left(\frac{\partial\Omega_y}{\partial x}\right)^2 + \tfrac{1}{2}A_{21}^{0}\left(\frac{\partial\Omega_x}{\partial y}\right)^2 + \tfrac{1}{2}A_{11}^{0}\left(\frac{\partial\Omega_x}{\partial x}\right)^2 \\ &\quad + \tfrac{1}{2}B_1^{0}\left(\frac{\partial\Omega_z}{\partial x}\right)^2 + \tfrac{1}{2}B_2^{0}\left(\frac{\partial\Omega_z}{\partial y}\right)^2 + \tfrac{1}{2}B_3^{0}\left(\frac{\partial\Omega_z}{\partial z}\right)^2 \\ &\quad + B_{13}^{0}\frac{\partial\Omega_z}{\partial z}\frac{\partial\Omega_z}{\partial x} + C_1^{0}\frac{\partial\Omega_x}{\partial x}\frac{\partial\Omega_z}{\partial x} + C_2^{0}\frac{\partial\Omega_x}{\partial y}\frac{\partial\Omega_z}{\partial y}. \end{aligned} \tag{3 b}$$

Since our definition of the elastic free energy density is given in a standard quadratic form, there is a slightly different definition of the constants A_{11} $(= -\tfrac{1}{2}A_{11}^{0})$ and C_1 $(= -C_1^{0})$ in equation (3 *a*) from that of the Orsay Group. In order to see that our equation (3 *a*), apart from the differences in the terms A_{11} and C_1 mentioned previously, is in fact identical to equation (3 *b*) in the Orsay Group's notation we must note the relations

$$-\tfrac{1}{2}\operatorname{div}[(\operatorname{div}\mathbf{a})\mathbf{a}]=\left(\frac{\partial\Omega_x}{\partial x}\right)^2+\frac{\partial\Omega_x}{\partial y}\frac{\partial\Omega_y}{\partial x} \qquad (4a)$$

$$\operatorname{div}[\mathbf{a}\operatorname{div}\mathbf{c}-(\nabla\mathbf{c})\mathbf{a}]=\frac{\partial\Omega_z}{\partial y}\frac{\partial\Omega_y}{\partial x}+\frac{\partial\Omega_x}{\partial x}\frac{\partial\Omega_z}{\partial x} \qquad (4b)$$

which allow us to transform (neglecting surface energies) the two terms $(\partial\Omega_x/\partial x)^2$ and $(\partial\Omega_x/\partial x)(\partial\Omega_z/\partial x)$ appearing in the Orsay Group's version of the elastic energy into a form corresponding to the one which we have written in our approach.

2.2. *Tilt angle dependence of the elastic constants*

The temperature dependence of the elastic constants has two contributions, one due to the temperature dependence of the forces by which the particles of the system interact and one due to the tilt angle dependence. The latter dependence has been discussed by Dahl and Lagerwall [8]; we repeat their arguments briefly here and advance the discussion a little further. The elastic energy of the system must be invariant if we change the tilt θ to $-\theta$. If at the same time we keep the layer normal **a** unaffected this implies that the elastic energy given by equation (3 *a*) must be invariant if we simultaneously make the changes $\mathbf{c}\to-\mathbf{c}$ and $\mathbf{b}\to-\mathbf{b}$. This implies that the coefficients A_{12}, A_{21}, A_{11}, B_1, B_2 and B_3 must be even functions of θ while the coefficients B_{13}, C_1 and C_2 must be odd functions of θ. In the smectic A phase (where $\theta=0$) only the A constants should remain (we can still make layer deformations) while all of the other coefficients must vanish. Expanding the elastic coefficients in powers of θ we thus must be able to write the A coefficients in the form $A_i\approx K_i+\bar{A}_i\theta^2$ and the three coefficients B_1 to B_3 as $B_i\approx\bar{B}_i\theta^2$. The coefficient B_{13}, being odd in θ, cannot have a linear dependence in θ because of an inequality (see equation (8 *c*)) which is derived in § 2.3. However, we can allow the C coefficients to be linear in θ. Further, in the S_A phase the elastic energy is given by [1]

$$w_A=\tfrac{1}{2}K(\nabla\cdot\mathbf{a})^2. \qquad (5)$$

The elastic energy of the S_C phase reduces to

$$w=\tfrac{1}{2}[K_{12}(\mathbf{b}\cdot\nabla\times\mathbf{c})^2+K_{21}(\mathbf{c}\cdot\nabla\times\mathbf{b})^2+2K_{11}(\mathbf{b}\cdot\nabla\times\mathbf{c})(\mathbf{c}\cdot\nabla\times\mathbf{b})], \qquad (6a)$$

when the system approaches the S_A phase and we take the limit as $\theta\to0$ in equation (3 *a*). Moreover, the vector identity

$$(\nabla\cdot\mathbf{a})^2=(\mathbf{b}\cdot\nabla\times\mathbf{c})^2+(\mathbf{c}\cdot\nabla\times\mathbf{b})^2-2(\mathbf{b}\cdot\nabla\times\mathbf{c})(\mathbf{c}\cdot\nabla\times\mathbf{b}) \qquad (6b)$$

can be proved [10] to be valid for the three vectors **a**, **b** and **c**. By comparing equations (5) and (6) we notice that the constants K_i must fulfil the relations $K_{12}=K_{21}=-K_{11}\equiv K$ if the expressions we have for the S_A and S_C elastic energies are to coincide at $T_{S_CS_A}$. We are thus able to write down the tilt angle dependence of the elastic constants as

$$A_{12}=K+\bar{A}_{12}\theta^2,\quad A_{21}=K+\bar{A}_{21}\theta^2,\quad A_{11}=-K+\bar{A}_{11}\theta^2, \qquad (7a)$$

$$B_1=\bar{B}_1\theta^2,\qquad B_2=\bar{B}_2\theta^2,\qquad B_3=\bar{B}_3\theta^2, \qquad (7b)$$

$$B_{13}=\bar{B}_{13}\theta^3, \qquad C_1=\bar{C}_1\theta, \qquad C_2=\bar{C}_2\theta, \tag{7 c}$$

where the constants K, $\bar{A}_i$, $\bar{B}_i$ and $\bar{C}_i$ can be assumed only to be weakly temperature dependent.

2.3. *Inequalities for the elastic constants*

The values of the elastic constants are limited by a set of inequalities. These inequalities are already given in the original paper by the Orsay Group. However, there is a misprint in one of the inequalities given and in view of this we now sketch briefly the derivation of these inequalities. The elastic energy given by equation (3 *a*) is a quadratic form of five independent variables X_i, each of which corresponds to one possible independent distortion of the system. Each coefficient corresponding to one of the five pure deformations must be positive as such a deformation must produce a positive elastic energy if the undistorted flat layer is the stable equilibrium of the system. Concerning the cross coupling coefficients we reason as follows. Imagine we make a deformation of the system consisting of only two of the five eigen deformations. Let us, as an example, study the case of a pure layer deformation. The corresponding elastic energy can then be written as

$$2w=A_{12}X_1^2+A_{21}X_2^2+2A_{11}X_1X_2,$$

where we have introduced the variables X_1 and X_2 as abbreviations for the corresponding vector operators. Such a quadratic form can be written in matrix notation as $2w=\mathbf{X}^T\mathbf{A}\mathbf{X}$ where

$$\mathbf{X}=\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

is a state vector and $\mathbf{A}$ is the symmetric matrix

$$\begin{pmatrix} A_{12} & A_{11} \\ A_{11} & A_{21} \end{pmatrix}.$$

The given quadratic form is positive definite if, and only if, the leading minors of $\mathbf{A}$ (that is, A_{12} and the determinant of $\mathbf{A}$) are positive. We thus conclude that the inequality $A_{12}A_{21}-A_{11}^2>0$ must be fulfilled. By making an analogous argument for the other three cross coupling terms in equation (3 *a*) we finally arrive at the inequalities

$$A_{12}, A_{21}, B_1, B_2, B_3>0, \tag{8 a}$$

$$A_{12}A_{21}-A_{11}^2>0, \tag{8 b}$$

$$B_1B_3-B_{13}^2>0, \tag{8 c}$$

$$B_2A_{12}-C_1^2>0, \tag{8 d}$$

$$B_2A_{21}-C_2^2>0, \tag{8 e}$$

which the elastic constants must fulfil, where the inequality in equation (8 *d*) is different from the corresponding one given by the Orsay Group. More complicated inequalities may be derived in the same manner by studying deformations containing three of the basic deformations. By introducing the tilt angle dependence of the A constants (see equation (7 *a*)) we can reformulate inequality (8 *b*) as

$$\bar{A}_{12}+\bar{A}_{21}+2\bar{A}_{11}>0. \tag{9}$$

Moreover, it is clear from equation (5) that the relation

$$K>0 \tag{10}$$

must also be valid.

2.4. *Summary*

To study the elastic properties of the S_C phase we need a nine term elastic energy which we take to be that given by equation (3 *a*). The elastic constants of the system must fulfil the inequalities given by equations (8) to (10). The main temperature dependences of the elastic constants enter through their tilt angle dependence and to lowest order in the tilt we can express this according to equations (7). We also note that the tilt angle independent parts of the A constants are related, as given by equations (7 *a*).

3. The wedge problem—elastic energy and stability of a S_C liquid crystal in a wedge

3.1. *Geometrical arrangement of the smectic layers*

For the calculations in this paper we consider a S_C liquid crystal confined in a wedge where we have arranged the smectic planes as shown in figure 3. We assume the liquid crystal to be bounded by two glass plates at an angle β. We further assume that the surfaces of the glass plates have been treated in such a way that the director will be forced to point in a certain direction (strong anchoring) which we choose to be parallel to the plates. In order to describe the problem we find it convenient to introduce a cylindrical polar coordinate system (r, α, z) where r measures the distance radially outwards, α is the polar angle and z is the coordinate of the direction for which we assume translational symmetry of the system (we only impose boundary conditions on the bounding plates which are assumed to be infinitely extended in the z direction). The basis vectors of the coordinate system are such that $\hat{\mathbf{r}}$ will coincide with the layer normal while the smectic planes are parallel to the αz surface with $\hat{\boldsymbol{\alpha}}$ always pointing in the direction in which the layers are bending and $\hat{\mathbf{z}}$ being parallel to the axis of the wedge. In order to describe the c-director we introduce an angle ϕ, which is defined as the angle between the c-director and the z axis, taking ϕ positive as indicated in figure 3. With these assumptions and the definition of $\mathbf{b}$ given by equation (1) we can write down the following ansatz for $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ expressed in cylindrical coordinates

$$\mathbf{a}=\hat{\mathbf{r}}, \tag{11 a}$$

$$\mathbf{b}=-\hat{\boldsymbol{\alpha}}\cos\phi+\hat{\mathbf{z}}\sin\phi, \tag{11 b}$$

$$\mathbf{c}=\hat{\boldsymbol{\alpha}}\sin\phi+\hat{\mathbf{z}}\cos\phi, \tag{11 c}$$

where the only degree of freedom assumed for the system is $\phi=\phi(r,\alpha)$.

We now must investigate the circumstances under which we can achieve the configuration of the smectic layers assumed in figure 3. We require the smectic layers to be parts of cylinders with a common axis coinciding with the centre of the wedge. For a given boundary condition (i.e. the director is assumed to adopt a prescribed direction $\mathbf{n}_b$ at the boundary) and for a given tilt θ, the direction in which the layer normal can point at the boundary is restricted by the relation

$$\sin\theta=|\mathbf{a}\times\mathbf{n}_b|. \tag{12}$$

By assuming $\mathbf{n}_b$ to be parallel to the bounding plates and letting the preferred direction of the director make an angle θ with respect to $\hat{\mathbf{r}}$, i.e. suppose the boundary condition to be

$$\mathbf{n}_b = \hat{\mathbf{r}} \cos\theta + \hat{\mathbf{z}} \sin\theta, \tag{13}$$

we will achieve the configuration of figure 3 with the c-director varying in space, where $\mathbf{c} = \hat{\mathbf{z}}$ ($\phi = 0$) is one possible equilibrium configuration of the system.

We have given one boundary condition for which the assumption of the smectic layers pictured in figure 3, i.e. $\mathbf{a} = \hat{\mathbf{r}}$, is compatible. It is easy to verify that the constraint $\nabla \times \mathbf{a} = 0$ is fulfilled in this case. There is, however, one distortion of this configuration which would allow $\mathbf{a}$ to relax while maintaining the constraint $\nabla \times \mathbf{a} = 0$. This is the case where the curvature of the smectic layers is decreased in such a way that the common axis of the cylinders moves to the left in figure 3, while remaining in the plane of symmetry of the wedge. This is, however, not allowed if we assume strong anchoring of the molecules at the bounding plates. We thus conclude that the boundary condition which we have introduced forces the system to be geometrically stable according to the definition in § 1, i.e. the layers are forced to remain unchanged due to an interplay between the constraint $\nabla \times \mathbf{a} = 0$ and the boundary conditions.

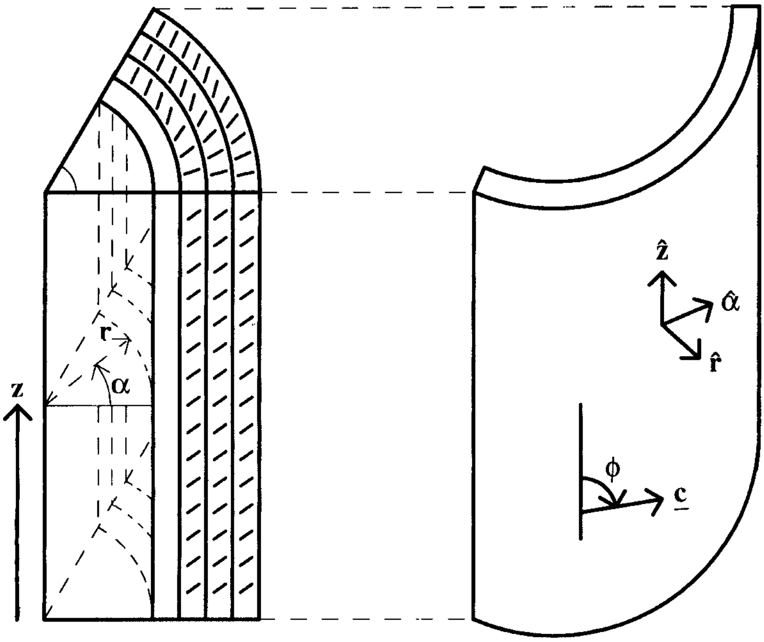

Figure 3. Set up of the wedge configuration. In the left hand part of the figure we see the actual wedge and the curved smectic layers, while the right hand part visualizes one smectic layer as seen when looking into the wedge. The coordinate system used is a cylindrical one for which the basis vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\alpha}}$ and $\hat{\mathbf{z}}$ are oriented as shown in the right hand part of the figure. The smectic planes are assumed to be parts of concentric cylinders with the common axis coinciding with the centre of the wedge and the layer normal $\mathbf{a}$ pointing in the r direction. The angle ϕ describing the c-director is defined as the angle between the z axis and the c-director as shown in the figure.

3.2. *The wedge elastic energy*

We now give the elastic energy density for the general case depicted in figure 3 for a S_C liquid crystal contained in a wedge. Upon substituting the ansatz of equations (11) into equation (3 *a*) and by supposing $\phi=\phi(r,\alpha)$ we find the elastic free energy density to be

$$\begin{aligned} w=&\tfrac{1}{2}(1/r^2)[A_{12}\sin^4\phi+A_{21}\cos^4\phi-2A_{11}\sin^2\phi\cos^2\phi] \\ &+\tfrac{1}{2}(1/r^2)[B_1\sin^2\phi+B_2\cos^2\phi]\phi_{,\alpha}^2+\tfrac{1}{2}B_3\phi_{,r}^2 \\ &+(1/r)B_{13}\sin\phi\,\phi_{,r}\phi_{,\alpha}+(1/r^2)[C_1\sin^2\phi-C_2\cos^2\phi]\cos\phi\,\phi_{,\alpha}. \end{aligned} \tag{14}$$

In this equation the notation $\phi_{,r}$ and $\phi_{,\alpha}$ denotes the partial derivatives of ϕ with respect to r and α, respectively. The first bracket of terms in equation (14) is connected to the bending of the layers and can be rewritten as

$$w_{\text{layer}}=(1/(2r^2)[(A_{12}+A_{11})\sin^4\phi+(A_{21}+A_{11})\cos^4\phi-A_{11}]. \tag{15}$$

We thus see that apart from the constant contribution

$$-A_{11}/(2r^2)=(K-\bar{A}_{11}\theta^2)/2r^2,$$

the A constants always enter the elastic energy by the two combinations $A_{12}+A_{11}$ and $A_{21}+A_{11}$. Introducing the tilt dependence of these constants according to equation (7 *a*) we can write

$$A_{12}+A_{11}=(\bar{A}_{12}+\bar{A}_{11})\theta^2, \tag{16 a}$$

$$A_{21}+A_{11}=(\bar{A}_{21}+\bar{A}_{11})\theta^2, \tag{16 b}$$

where we observe that the tilt independent part K of the A constants exactly cancels in the ϕ dependent parts of w_{layer}.

To calculate the equilibrium configuration of the system we shall minimize the total elastic energy

$$W=\int_V w\,\mathrm{d}V=\iiint_V wr\,\mathrm{d}r\,\mathrm{d}\alpha\,\mathrm{d}z.$$

We define the effective elastic energy density $w_e=wr$ which, if we introduce the tilt dependence of the elastic constants (see equations (7) and (16)), can be rewritten as

$$\begin{aligned} w_e=rw=&-A_{11}/(2r)+1/(2r)[(\bar{A}_{12}+\bar{A}_{11})\sin^4\phi+(\bar{A}_{21}+\bar{A}_{11})\cos^4\phi]\theta^2 \\ &+1/(2r)[\bar{B}_1\sin^2\phi+\bar{B}_2\cos^2\phi]\theta^2\phi_{,\alpha}^2+\frac{r}{2}\bar{B}_3\theta^2\phi_{,r}^2 \\ &+\bar{B}_{13}\theta^3\sin\phi\,\phi_{,r}\phi_{,\alpha}+(1/r)[\bar{C}_1\sin^2\phi-\bar{C}_2\cos^2\phi]\theta\cos\phi\,\phi_{,\alpha}. \end{aligned} \tag{17}$$

The effective elastic free energy density w_e in this equation is the expression which we shall use in our calculations.

The starting configuration for these calculations is that given by $\mathbf{a}=\hat{\mathbf{r}}$ and $\mathbf{c}=\hat{\mathbf{z}}$ ($\phi=0$). However, from equation (17) we note that for a constant ϕ the elastic energy of the system is minimized only for certain values of ϕ. The relevant quantity to minimize in order to calculate the stable equilibrium angle ϕ_0 is clearly

$$w_e^{\text{layer}}(\phi)=1/(2r)[(\bar{A}_{12}+\bar{A}_{11})\sin^4\phi+(\bar{A}_{21}+\bar{A}_{11})\cos^4\phi]\theta^2. \tag{18}$$

The minimization of w_e^{layer} corresponds to calculating the stable equilibrium angle ϕ_0 in the case when the system is not subject to any boundary conditions. As we have discussed the boundary conditions are essential for stabilizing the formation of the cylindrical structure of the layers within the wedge. If, however, we study a system for which the layers form concentric cylinders the equilibrium angle ϕ_0 obtained by minimizing w_e^{layer} is feasible and is obtained from $dw_e^{\text{layer}}/d\phi=0$ as

$$\sin\phi\cos\phi[(\bar{A}_{12}+\bar{A}_{11})\sin^2\phi-(\bar{A}_{21}+\bar{A}_{11})\cos^2\phi]=0. \tag{19}$$

Before we can conclude which of the possible solutions of this equation corresponds to a minimum of the elastic energy of the system we note that inequality (9) permits at most one of the two quantities $\bar{A}_{12}+\bar{A}_{11}$ and $\bar{A}_{21}+\bar{A}_{11}$ to be negative. This implies that, regarding the stability of the system, there are three different cases which can occur. (Note that due to the symmetry of the system the solutions $\phi=0$ and $\phi=\pi$ represent the same configuration; the same is true for $\phi=\pi/2$ and $\phi=3\pi/2$. We do not consider these trivial solutions in the later discussion.)

Case 1: $\bar{A}_{12}+\bar{A}_{11}>0$ and $\bar{A}_{21}+\bar{A}_{11}>0$

The stable equilibrium angle is given by

$$\tan^2\phi_0=(\bar{A}_{21}+\bar{A}_{11})/(\bar{A}_{12}+\bar{A}_{11}) \tag{20}$$

while the solutions $\phi=0$ and $\phi=\pi/2$ correspond to maxima of the elastic energy of the system.

Case 2: $\bar{A}_{12}+\bar{A}_{11}>0$ and $\bar{A}_{21}+\bar{A}_{11}<0$

The stable equilibrium angle is given by $\phi_0=0$ while the solution $\phi_0=\pi/2$ corresponds to a maximum of the elastic energy of the system.

Case 3: $\bar{A}_{12}+\bar{A}_{11}<0$ and $\bar{A}_{21}+\bar{A}_{11}>0$

The stable equilibrium angle is given by $\phi_0=\pi/2$ while the solution $\phi=0$ corresponds to a maximum of the elastic energy of the system.

Returning to the wedge problem we now conclude that if we examine a wedge where we have strong anchoring of the director at the boundaries, which conflicts with the stable equilibrium angle ϕ_0, we should not expect the configuration $\phi=$ constant to be stable under all circumstances. What can happen is that the c-director rotates in order to seek out the angle ϕ_0. This will create a non-uniform configuration with $\phi_{,\alpha}\neq 0$ which will be associated with the elastic energy

$$[\bar{B}_1\sin^2\phi+\bar{B}_2\cos^2\phi]\theta^2\phi_{,\alpha}^2/2r.$$

If the wedge angle is small enough the elastic energy associated with the c-director rotation will be larger than the gain of layer energy and the configuration $\phi=$ constant will be stable. For a large enough wedge angle β on the other hand a transition into a state with a non-uniform ϕ will be energetically favourable. We thus expect a transition where the control parameter is the wedge angle β. In the next section we obtain the governing differential equations of the system and calculate the critical wedge angle β for which the configuration $\mathbf{c}=\hat{\mathbf{z}}$ $(\phi=0)$ becomes unstable against rotations of the c-director.

4. Frederiks transition in a wedge—calculation of thresholds and stability considerations

We assume throughout this paper that the dielectric anisotropy ε_a of the liquid crystal which is studied is positive. Preparing such a system in the wedge configuration of figure 3, we can induce a Frederiks transition by applying a voltage across the bounding plates of the wedge. In this section we derive the threshold of this transition.

4.1. *The electric field free energy density*

The electric free energy density which arises when an electric field is applied over a liquid crystal is given by [2]

$$w^{\varepsilon}=-\tfrac{1}{2}\varepsilon_a\varepsilon_0(\mathbf{n}\cdot\mathbf{E})^2, \tag{21}$$

where ε_a is the dielectric anisotropy of the liquid crystal and ε_0 is the permittivity of free space. In the geometry of figure 3 the director is given by

$$\mathbf{n}=\hat{\mathbf{r}}\cos\theta+\hat{\boldsymbol{\alpha}}\sin\theta\sin\phi+\hat{\mathbf{z}}\sin\theta\cos\phi. \tag{22}$$

The unperturbed configuration is assumed to correspond to $\phi=0$ and thus the corresponding director configuration is given by

$$\mathbf{n}=\hat{\mathbf{r}}\cos\theta+\hat{\mathbf{z}}\sin\theta.$$

In order to induce a Frederiks transition we shall apply an electric field $\mathbf{E}=E\hat{\boldsymbol{\alpha}}$. This field will be achieved if we apply a voltage U across the bounding plates in which case we obtain

$$\mathbf{E}=\frac{U}{r\beta}\hat{\boldsymbol{\alpha}}. \tag{23}$$

The electric free energy density for a general configuration of the system is now given by equations (21) to (23) as

$$w^{\varepsilon}=-\tfrac{1}{2}\varepsilon_a\varepsilon_0\left(\frac{U}{r\beta}\right)^2\sin^2\theta\sin^2\phi. \tag{24}$$

When minimizing the total free energy we integrate w^{ε} over the volume of the system. As in the derivation of the effective elastic free energy density of equation (17) a scaling factor r will enter the electric free energy density. We thus introduce the effective electric free energy density

$$w_e^{\varepsilon}=-\tfrac{1}{2}\varepsilon_a\varepsilon_0\frac{U^2}{r\beta^2}\sin^2\theta\sin^2\phi. \tag{25}$$

4.2. *The Euler–Lagrange equations—calculation of the Frederiks threshold*

In order to minimize the total free energy of the system we must find the configuration $\phi(r,\alpha)$ which minimizes the integral

$$W=\int\bar{w}(\phi)\,dr\,d\alpha\,dz,$$

where, using equations (17) and (25)

$$\bar{w}=w_e+w_e^{\varepsilon}.$$

This problem is solved by the Euler–Lagrange equation

$$\frac{\partial}{\partial r}\frac{\partial \bar{w}}{\partial(\phi_{,r})}+\frac{\partial}{\partial\alpha}\frac{\partial \bar{w}}{\partial(\phi_{,\alpha})}-\frac{\partial \bar{w}}{\partial\phi}=0. \tag{26}$$

By substituting equations (17) and (25) into this equation we derive the governing equation of the system

$$\begin{aligned} &r^2\bar{B}_3\phi_{,rr}+(\bar{B}_1\sin^2\phi+\bar{B}_2\cos^2\phi)\phi_{,\alpha\alpha}+\tfrac{1}{2}(\bar{B}_1-\bar{B}_2)\sin(2\phi)(\phi_{,\alpha})^2\\ &+\bar{B}_{13}r\theta\cos\phi\,\phi_{,\alpha}\phi_{,r}+2\bar{B}_{13}r\theta\sin\phi\,\phi_{,\alpha r}+\bar{B}_3 r\phi_{,r}\\ &+2[(\bar{A}_{21}+\bar{A}_{11})\cos^2\phi-(\bar{A}_{12}+\bar{A}_{11})\sin^2\phi]\sin\phi\cos\phi\\ &+\varepsilon_a\varepsilon_0\left(\frac{U}{\beta}\right)^2\sin\phi\cos\phi=0. \end{aligned} \tag{27}$$

In deriving this result we have made the approximation $\sin\theta\approx\theta$ in the term related to the electric field in order to observe that the θ dependence actually disappears for most terms. The θ dependence only remains in the $\bar{B}_{13}$ terms. These terms may, therefore, be neglected if we study the system relatively close to the transition to the S_A phase.

As we are interested in the system close to the Frederiks threshold we assume ϕ to be small; linearizing equation (27) we obtain

$$\bar{B}_3 r^2\phi_{,rr}+\bar{B}_2\phi_{,\alpha\alpha}+\bar{B}_3 r\phi_{,r}+2(\bar{A}_{21}+\bar{A}_{11})\phi+\varepsilon_a\varepsilon_0\left(\frac{U}{\beta}\right)^2\phi=0. \tag{28}$$

As no boundary conditions are assumed to be applied in the r direction, the simplest solution we can achieve is that for which $\phi_{,r}$ is zero, i.e. $\phi(r,\alpha)=\phi(\alpha)$. Equation (28) then reduces to

$$\bar{B}_2\phi_{,\alpha\alpha}+2(\bar{A}_{21}+\bar{A}_{11})\phi+\varepsilon_a\varepsilon_0\left(\frac{U}{\beta}\right)^2\phi=0. \tag{29}$$

In order to fulfil the boundary conditions ($\phi=0$ for $\alpha=0,\beta$) we make the ansatz

$$\phi(\alpha)=\phi_m\sin\left(\frac{\pi}{\beta}\alpha\right). \tag{30}$$

Inserting this ansatz into equation (29) we derive the Frederiks threshold as

$$\varepsilon_a\varepsilon_0 U_c^2=\pi^2\bar{B}_2-2\beta^2(\bar{A}_{21}+\bar{A}_{11}). \tag{31}$$

We see therefore that by varying the wedge angle β, we can determine the constants $\bar{B}_2$ and $\bar{A}_{21}+\bar{A}_{11}$ by measuring the Frederiks threshold for each value of β.

4.3. *Derivation of the Frederiks threshold by energy considerations*

In this section we sketch briefly how by an energy consideration we can derive the threshold of the Frederiks transition by a more direct method than that used in the previous section. Let us explicitly calculate the total free energy of the system when we make a small perturbation from the starting configuration $\phi=0$. The simplest perturbation we can make, fulfilling the boundary conditions is that given by equation (30), assuming ϕ_m to be a small quantity. The total free energy of the system is then

$$W=\int_{r_1}^{r_2}\int_0^{\beta}\int_{z_1}^{z_2}(w_e+w_e^{\varepsilon})\,dr\,d\alpha\,dz, \tag{32}$$

where w_e and w_e^{ε} are given by equations (17) and (25), respectively, and, in order to perform the integration, we further assume the system to be bounded in the r and z directions. Only keeping the terms to lowest order in ϕ_m, equations (17), (25), (30) and (32) give

$$W=(z_2-z_1)\ln\frac{r_2}{r_1}\left\{\tfrac{1}{2}A_{21}+\tfrac{1}{4}\phi_m^2\left[\bar{B}_2\left(\frac{\pi}{\beta}\right)^2-2(\bar{A}_{21}+\bar{A}_{11})-\varepsilon_a\varepsilon_0\left(\frac{U}{\beta}\right)^2\right]\theta^2\right\}\beta, \quad (33)$$

where we again have approximated $\sin\theta$ by θ. The system will lower its energy by adopting a finite ϕ_m if the factor multiplying ϕ_m^2 in equation (33) is negative. This will be the case of the applied voltage U is large enough and from equation (33) we immediately derive the expression of the Frederiks threshold. By comparing the result thus obtained with equation (31) we note that by this energy method we have simply rederived the previously obtained threshold for the transition.

4.4. *Stability of the assumed configuration—derivation of the critical wedge angle*

As we discussed at the end of § 3.2, the configuration $\mathbf{c}=\hat{\mathbf{z}}$ need not be a stable equilibrium in the wedge even under a zero electric field. We are now in a position to calculate the critical wedge angle β_c, which is the largest wedge angle for which the assumed configuration is stable against distortions of the c-director. From equation (33) we find the total free energy of the system in the absence of an electric field to be

$$W=D_1+D_2\phi_m^2[\bar{B}_2\pi^2-2(\bar{A}_{21}+\bar{A}_{11})\beta^2], \quad (34)$$

where D_1 and D_2 are constants which are irrelevant to the present discussion. This result expresses the fact that there is competition between two contributions to the elastic energy of the system. The $\bar{B}_2$ term corresponds to the energy associated with a c-director which is non-uniform with respect to the smectic layers. If the term $\bar{A}_{21}+\bar{A}_{11}$ is positive then it will decrease the elastic energy for $\phi_m\neq 0$. This term originates from the fact that for this sign of $\bar{A}_{21}+\bar{A}_{11}$ the elastic energy associated with the bending of the smectic layers (see equation (18)) is not at a minimum for the configuration $\phi=0$. If the original configuration should be stable the total free energy of the system must increase with ϕ_m and thus the coefficient multiplying ϕ_m^2 in equation (34) must be positive. This leads to a limitation for β given by

$$\beta<\pi\surd[\bar{B}_2/2(\bar{A}_{21}+\bar{A}_{11})]. \quad (35)$$

As we discussed in § 3.2, we do not know the sign of $\bar{A}_{21}+\bar{A}_{11}$. If $\bar{A}_{21}+\bar{A}_{11}$ is negative the configuration $\phi=0$ is a stable equilibrium and inequality (35) becomes irrelevant. If on the other hand $\bar{A}_{21}+\bar{A}_{11}$ is positive the inequality (35) gives a limitation of how large the wedge angle β can be if we want the configuration $\phi=0$ to be stable. However, until the constants $\bar{B}_2$ and $\bar{A}_{21}+\bar{A}_{11}$ have been measured it is not possible to decide whether or not inequality (35) provides a real limitation to the magnitude of the feasible wedge angle. (If the critical wedge angle is found to be larger than 2π the limitation of equation (35) becomes irrelevant.)

5. Discussion

As is shown in figure 2, a general elastic deformation of a S_C liquid crystal can be regarded to be composed of five different distortions, all of which can be made independently from each other. Two of these distortions represent deformations of the smectic layers (see figures 2 (*b*) and (*c*)) while the other three represent rotations of the c-director (see figures 2 (*d*)–(*f*)). The bulk elastic free energy density of the system can be expressed in terms of these five deformations and will, altogether, consist of nine terms. Of these nine terms five correspond to the eigen deformations described previously, while the other four represent the cross-coupling terms which are allowed by the symmetry of the system. To each of these five eigen deformations we can assign a vector operator, which is also given in figure 2. By expressing these vector operators in terms of the two unit vectors **b** and **c** (see figure 1) rather than **a** and **c** (which at first sight might be thought to be more natural) we have shown how we can achieve a particularly simple expression for each operator. The elastic free energy density of the system can be written as a quadratic form in terms of the five vector operators (see equation (3 *a*)), which closely resembles that given by Rapini [10]. By the use of the operator identities (see equation (4)) we have also shown that, neglecting surface energies, our form of the elastic energy supplements and extends the one previously derived by the Orsay Group [1]. The nine elastic constants which are defined through equation (3 *a*) must fulfil certain restrictions. Firstly, by the symmetry of the system, we expect a tilt angle dependence of the elastic constants as given by equations (7). Moreover, as the elastic energy represents an expansion around a stable equilibrium, the quadratic form of equation (3 *a*) must be positive definite. Thus the elastic constants must fulfil the inequalities of equations (8) to (10). By demanding that the elastic energy of the S_C phase should reduce to that of the S_A phase (see equation (5)) in the limit as $\theta \rightarrow 0$ we have shown in § 2.2 that the tilt independent parts of the elastic constants describing the layer deformations (the A constants) must be related as stated by equations (7 *a*). For this reason, the inequality (8 *b*) can be replaced by two other inequalities, providing one inequality relating the tilt dependent parts $\bar{A}$ of these constants (see equation (9)) and one inequality merely stating that the tilt independent part K of these constants must be positive.

When preparing a sample of S_C liquid crystal the possible ways of arranging the smectic layers are restricted by the constraint $\nabla \times \mathbf{a} = 0$. By specifying the boundary conditions for the director and demanding the layer thickness to be constant, we find that most configurations we can achieve are geometrically stable, i.e. the smectic layers are forced to remain unchanged due to an interplay between the constraint $\nabla \times \mathbf{a} = 0$ and the boundary conditions. Thus most dynamic experiments performed on S_C liquid crystals only involve the rotation of the c-director and to the best of our knowledge no measurements of the elastic constants relating to the layer deformations have been reported in the literature. In a paper by Rapini [10] a series of Frederiks transitions for S_C liquid crystals with planar layers have been considered. In Rapini's paper the calculated thresholds generally contain all of the nine elastic constants included in the theory. However, the distortions of the smectic layers assumed in these calculations violate the constraint $\nabla \times \mathbf{a} = 0$. Thus the thresholds calculated by Rapini are more of academic interest and cannot be used to measure all of the nine elastic constants involved in the expression of the S_C elastic energy. As is also pointed out by Rapini, all transitions considered in his paper which involve a coupling of layer deformations and c-director rotations would, as soon as they have been initiated, involve layer compressions which demand a very large elastic energy. Thus the layer deformations will immediately be suppressed and the transition will continue as a pure c-director rotation. As a result of this Rapini calls such transitions ghost transitions, i.e. we can establish a theoretical threshold for them but we cannot achieve them in reality. For this reason the Frederiks transitions considered by Rapini will not permit us to measure any elastic constants other than those related to the c-director rotations.

In this paper we have shown how, by assuming the wedge geometry of figure 3 combined with the boundary conditions given by equation (13), we can design an experiment where the threshold of the Frederiks transition considered also involves the A constants despite the distortions of the system consisting only of pure c-director rotations. To our knowledge this is the first experimental set up suggested which would make it possible to extract any information about the A constants. The elastic free energy density in the wedge is given by equation (17). From this expression we see that the elastic constants which we can expect to deduce in this case (apart from the B constants) are the combinations $\bar{A}_{12}+\bar{A}_{11}$ and $\bar{A}_{21}+\bar{A}_{11}$. With the boundary conditions we have assumed we further see that by measuring the critical wedge angle (if it exists) given by equation (35) and by measuring the wedge angle dependence of the threshold for the Frederiks transition (see equation (31)) we will in both cases only gain information of the elastic constants $\bar{B}_2$ and $\bar{A}_{21}+\bar{A}_{11}$. If we desire information about the elastic constant $\bar{A}_{12}+\bar{A}_{11}$, equation (17) tells us that we should instead use a boundary condition which is compatible with an undistorted director configuration $\mathbf{c}=\hat{\boldsymbol{\alpha}}$ ($\phi=\pi/2$), i.e. we should confine the director in the $r\alpha$ plane at the boundaries, demanding it to make an angle θ with the plates. This boundary condition corresponds to

$$\mathbf{n}_b=\hat{\mathbf{r}}\cos\theta+\hat{\boldsymbol{\alpha}}\sin\theta. \tag{36}$$

Expanding the elastic free energy density around $\phi=\pi/2$ we can, in a similar way to that carried out in §4.4, derive the largest possible wedge angle for which this configuration is stable against rotations of the c-director (provided $\bar{A}_{12}+\bar{A}_{11}>0$) as

$$\beta<\pi\sqrt{[\bar{B}_1/2(\bar{A}_{12}+\bar{A}_{11})]}. \tag{37}$$

If the wedge angle does not exceed the critical angle then a Frederiks transition induced for this configuration would give information about the constants $\bar{B}_1$ and $\bar{A}_{12}+\bar{A}_{11}$. There is, however, one fundamental difficulty with performing this experiment. Generally, the Frederiks threshold is voltage dependent, implying that the critical electric field is inversely proportional to the thickness of the sample. However, since the effective thickness of the sample increases as we travel radially outwards from the centre of the wedge, we see that when applying a voltage U across the two plates, as for the transition discussed in § 4, the electric field will decrease inversely with r according to equation (22). Thus the transition will be induced over the whole sample simultaneously, a fact which justifies the assumption that we can neglect the r dependence of ϕ when deriving the threshold in § 4. Performing a Frederiks transition with the boundary conditions given by equation (36), however, implies that the electric field must be applied parallel to the z axis, i.e. $\mathbf{E}=E\hat{\mathbf{z}}$. Such a field does not scale as $1/r$ and we must, therefore, expect the transition to start at the outer boundary of the wedge. The twist elasticity between the layers, i.e. the B_3 terms will then stabilize the system and the mathematical treatment of the problem becomes more involved. Allowing ϕ to be a function of r also, we can solve the problem in terms of Bessel functions. This approach does not, however, allow us to determine a convenient expression for the threshold in terms of a given set of constants. We can circumvent this difficulty by only filling the space of the wedge between R and $R+\delta R$, demanding $\delta R\ll R$. In this case we can derive an approximate solution of the problem by assuming $r\approx R$ over the whole sample, i.e. we neglect the twist elasticity between the layers (the B_3

terms) in the governing equations. From these considerations we derive the Frederiks threshold of the system as

$$\varepsilon_a\varepsilon_0\langle d\rangle E_c^2=\pi^2\bar{B}_1-2\beta^2(\bar{A}_{12}+\bar{A}_{11}), \tag{38}$$

where we have introduced the average sample thickness

$$\langle d\rangle=(R+\delta R/2)\beta.$$

From equation (38) it can be seen that by only partially filling the wedge we can achieve a transition where the influence of the twist elasticity between layers can be neglected to a reasonable degree of accuracy. We would then be able to design an experiment which allows us to measure the elastic constant $\bar{A}_{12}+\bar{A}_{11}$ also. Nevertheless, the requirement $\delta R\ll R$ combined with the need to keep the wedge angle as large as possible while the average thickness $\langle d\rangle$ should not exceed feasible values, may cause geometrical problems concerning the realizability of such an experiment.

The temperature independent part K of the A constants has not yet been determined experimentally. As we have noted, the constant K does not enter any of the calculated thresholds and so it cannot be determined from any of our suggested experiments. If, however, we assume a boundary condition for which the director is confined parallel to the bounding plate but is allowed to point in any direction parallel to the plate, it may be possible to measure K in the following way. For the new boundary conditions proposed, the wedge configuration of figure 3 is no longer geometrically stable because the cylindrical layers can increase their radius by rotating the **n**-director thus causing a non uniform **c**-director configuration $\mathbf{c}(\alpha)$ within each layer. The system thus exhibits two extreme configurations:

(i) The configuration of figure 3, $\mathbf{c}=\hat{\mathbf{z}}$. Here all of the elastic energy is due to the bending of the layers and will be given by the term $(K-\bar{A}_{11}\theta^2)/(2r)$ in equation (17).

(ii) Flat layers, $\mathbf{c}(\alpha)$ varying with $\delta\phi\neq 0$ between the plates. Here all the elastic energy is due to **c**-director variations.

Analysing this situation would make it possible to derive a threshold for switching between the two extreme situations, the expression of which would contain K. The experimental determination of K would thus be possible from such a type of experiment.

To conclude, in this paper we have demonstrated how, by performing experiments in suitable geometries, we can gain information about the elastic constants which are connected to the deformations of the layers of a S_C liquid crystal, constants which are normally difficult to determine experimentally. The experiments which we have suggested may be difficult to realize for technical reasons, but seem to be among the most straightforward of those which can be proposed where the elastic constants of interest will appear in the analysis.

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